The electron thermal propagator at $p \gg T$: An entire function of p_0

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The retarded electron propagator $S_R(p_0, \mathbf{p})$ at high momentum $p \gg T$ was shown by Blaizot and Iancu to be an entire function of complex p_0. In this paper a specific form for $S_R(p_0, \mathbf{p})$ is obtained and checked by showing that its temporal Fourier transform $S_R(t, \mathbf{p})$ has the correct behavior at large t. Potential infrared and collinear divergences from the emission of soft photons do not occur.

I. INTRODUCTION

High temperature QCD is, in one respect, simpler than high temperature QED. In both theories the damping rates for charged particles at very high momentum, $p \gg T$, have infrared singularities at the mass shell when computed in perturbation theory using Braaten-Pisarski resummation [1]. The divergence comes from the emission and absorption of quasistatic, transverse gauge bosons. In QCD this apparent infrared divergence in the quark and gluon damping rates is cut off by the nonperturbative magnetic screening mass for transverse gluons and this gives finite damping rates [2]. For quarks the damping rate γ determines the location of the pole in the retarded propagator $s_R(p_0, p)$:

$$s_R^{\rm quark}(p_0, p) = \frac{1}{p_0 - p + i\gamma/2}.$$

The Fourier transform gives the time-dependent retarded propagator

$$s_R^{\rm quark}(t, p) = -i\theta(t)e^{-ipt}\exp[-\gamma t/2]$$

In QED there is no magnetic screening mass and consequently the lowest order electron damping rate is infrared divergent [1, 3, 4, 5, 6]. A direct analysis of the time dependent retarded propagator $s_R(t, p)$ produced surprising results: the time dependent propagator is not infrared divergent and at large times, $\omega_p t \gg 1$, it falls more rapidly than an exponential

$$s_R(t, p) \to -ie^{-ipt}\exp\Big\{-\alpha T t\big[\ln(\omega_p t) + C\big]\Big\}. \quad (1.1)$$

Here $\alpha = g^2/4\pi$ is the fine structure constant and $\omega_p = gT/3$ is the plasma frequency. (It is customary to use g for the electric charge so as not to confuse it with the base of the natural logarithm e.) Blaizot and Iancu [7, 8, 9] obtained this result using a functional integral version of the Bloch-Nordsieck approximation. This resums an infinite class of Feynman graphs with no fermion loops, except for the Braaten-Pisarksi hard-thermal-loop correction to the quasistatic transverse photon propagator. Wang, Boyanovsky, de Vega, and Lee obtained the same result by using the dynamical renormalization group to improve the lowest order result [10]. Boyanovsky et at [11] found the same behavior for the retarded propagator of the scalar particle in scalar QED. Because $s_R(t, p)$ falls faster than an exponential, its Fourier transform $s_R(p_0, p)$ has some very unusual properties. Blaizot and Iancu proved the following about $s_R(p_0, p)$: (1) It is an entire function of p_0; (2) It vanishes in the limit ${\rm Im}\, p_0 \to +\infty$; (3) It diverges in the limit ${\rm Im}\, p_0 \to -\infty$. Specifically, at $p_0 = p - i\zeta$ it is pure imaginary and bounded from below:

$$\zeta \to \infty: \qquad \big|s_R(p - i\zeta, p)\big| > \frac{1}{\alpha T}\exp\Big(a\exp\Big[\frac{\zeta}{\alpha T}\Big]\Big). \quad (1.2)$$

This asymptotic behavior was deduced in [8] using $C = 0$ and the constant a was shown to have the value $a = 3g\exp(-1)/4\pi$. However, revising the argument of [8] for a non-zero value of C gives

$$a = \frac{3g}{4\pi}\exp[-C - 1] \quad (1.3)$$

A propagator that is an entire function of p_0 has no poles, no branch points, and no essential singularities.

The purpose of this paper is to find the function $s_R(p_0, p)$ whose Fourier transform has the asymptotic behavior in Eq. (1.1). The physical approximations which lead to Eq. (1.1) are valid in the region $p_0 \approx p$ [7, 8, 9, 10, 11] and it is only this region that is investigated here. Sec. II shows that the Bloch-Nordsieck propagator $s_R(t, p)$ has the very unusual property of being an entire function of complex t. In Sec. III only the leading term in the asymptotic expansion is employed, viz.

$$-i\frac{\sqrt{2\pi a}}{\alpha T}\exp\Big[i\,\frac{p_0 - p}{2\alpha T} + a\exp\Big[i\,\frac{p_0 - p}{\alpha T}\Big]\Big]. \quad (1.4)$$

This approximation for $s_R(p_0, p)$ works surprisingly well. It is checked by showing that the Fourier transform over energies $p_0 \approx p$ agrees with Eq. (1.1) at large t. The motivation for the ansatz, Eq. (1.4), is given in Appendix B. There a complete asymptotic expansion of $s_R(p_0, p)$ is obtained. The first term of this expansion is Eq. (1.4). Sec. IV shows how the absence of singularities in the electron propagator imply that in hard scattering processes, there will be no infrared or collinear divergences and the quantitative contribution of these regions is numerically small. Sec. V discusses some further consequences.

II. COMMENT ON COMPLEX TIME

The logarithmic dependence on t in Eq. (1.1) does not imply that there is a branch cut for negative t. The full

Bloch-Nordsieck calculation of Blaizot and Iancu [7, 8, 9] gives

$$s_R(t,p) = -i\theta(t)e^{-ipt}\exp\big[-\alpha T t\, f(\omega_p t)\big],$$

where $f(z)$ is given by

$$f(z) = C - \gamma + \frac{1-e^{-z}}{z} + \int_0^z ds\,\frac{1-e^{-s}}{s}.$$

This representation shows that $f(z)$ is analytic everywhere in the complex z plane. (See also Appendix B.) Thus the retarded propagator with $\theta(t)$ omitted is analytic everywhere in the complex t plane.

Analyticity in t is very unusual. Appendix A shows that in general the retarded propagator (without the $\theta(t)$) is the difference of two functions: one, $S_>(t,p)$, is analytic in the open strip $-\beta < \mathrm{Im}\,(t) < 0$; the other $S_<(t,p)$ is analytic in the open strip $0 < \mathrm{Im}\,(t) < \beta$. Thus generally, the retarded propagator is only defined on the real t axis and is not analytic off-axis. Appendix A also shows that the analyticity in t of $s_R(t,p)$ does not improve on, or detract from, the validity of the Kubo-Martin-Schwinger condition [12], which requires $\mathcal{S}_>(t-i\beta,p) = -\mathcal{S}_<(t,p)$.

III. THE PROPAGATOR FOR $\mathbf{p_0 \approx p}$

This section will provide evidence that a good approximation to the retarded electron propagator in the region

$$p - \frac{\pi\alpha T}{2} < \mathrm{Re}\,p_0 < p + \frac{\pi\alpha T}{2}, \tag{3.1}$$

is the function

$$s_R(p_0,p) = -iN\exp\left[i\,\frac{p_0-p}{2\alpha T} + a\exp\left[i\,\frac{p_0-p}{\alpha T}\right]\right], \tag{3.2}$$

where $N = \sqrt{2\pi a}/\alpha T$. It is and entire function of p_0 and has the correct behavior anticipated in Eq. (1.2). The Fourier transform of Eq. (3.2) will be shown to reproduce Eq. (1.1) at large time. The motivation for the ansatz (3.2) is contained in Appendix B. That appendix contains a complete asymptotic expansion of $s_R(p_0,p)$ valid in the strip (3.1). Eq. (3.2) is first term in the expansion.

For any propagator, the spectral function (for real p_0) is

$$\rho(p_0,p) = i[s_R(p_0,p) - s_R(p_0,p)^*].$$

For the approximate propagator Eq. (3.2),

$$\begin{aligned}\rho(p_0,p) \;&=\; 2N\exp\left\{a\cos\left[\frac{p_0-p}{2\alpha T}\right]\right\}\\ &\quad\times\cos\left[\frac{p_0-p}{2\alpha T} + a\sin\left[\frac{p_0-p}{\alpha T}\right]\right].\end{aligned}$$

If a is small, this is positive in the range given by Eq. (3.1). For $|p_0-p| \ll \alpha\pi T$ the approximate behavior is

$$\rho(p_0,p) \approx 2Ne^a\left\{1 - \left(\frac{1}{8} + a + \frac{a^2}{2}\right)\left[\frac{p_0-p}{\alpha T}\right]^2\right\}.$$

A. Context for application

When the fermions are massless in QCD or QED, chirality and TCP invariance guarantees that the matrix structure of the retarded fermion propagator is

$$\begin{aligned}\mathcal{S}_R(p_0,\mathbf{p}) \;&=\; \frac{1}{2}(\gamma^0 - \gamma\cdot\hat{p})\,s_R(p_0,p)\\ &\quad-\; \frac{1}{2}(\gamma^0 + \gamma\cdot\hat{p})\,[s_R(-p_0^*,p)]^*.\end{aligned}$$

In applications, the fermion propagator appears in combination with a fermionic current of the form

$$\mathcal{J}_\lambda(x') = g\mathrm{Tr}\Big[e^{-\beta H}T\Big(A^\mu(x')\gamma_\mu\overline{\psi}_\lambda(x')\mathcal{O}(y_1,\ldots y_n)\Big)\Big]\frac{1}{Z},$$

where $Z = \mathrm{Tr}[\exp(-\beta H)]$ is the partition function. After projecting out the Dirac matrices from $\mathcal{S}_{\alpha\lambda}\mathcal{J}_\lambda$ a typical application the retarded propagator will produce the combination

$$\Psi(t,p) = -i\int_{-\infty}^{t} dt'\, s_R(t-t',p)\,J(t',p). \tag{3.3}$$

Ψ is the response at time t to the source function J acting at an earlier time t'. This can be expressed as an energy integral

$$\Psi(t,p) = \int\frac{dp_0}{2\pi}\,e^{-ip_0t}s_R(p_0,p)\widetilde{J}(p_0,p), \tag{3.4}$$

where $\widetilde{J}(p_0,p)$ is the Fourier transform of $J(t,p)$.

Example: For the conventional retarded propagator of the quark type (with a finite γ) the necessary energy integration is

$$\Psi(t,p) = \int\frac{dp_0}{2\pi}\frac{e^{-ip_0t}}{p_0 - p + i\gamma/2}\widetilde{J}(p_0,p)$$

If $\widetilde{J}(p_0,p)$ is analytic in p_0 in the lower-half plane and vanishes sufficiently rapidly as Im $p_0 \to -\infty$, then the p_0 contour can be closed in the lower-half plane and the result evaluated by Cauchy's theorem. The simplest way to insure that the source has these properties in p_0 is to insist that it vanish for times later than some t_f:

$$J(t',p) = \begin{cases} 0 & t' > t_f\\ \text{arbitrary} & t' < t_f.\end{cases} \tag{3.5}$$

The Fourier transform of the source is

$$\widetilde{J}(p_0,p) = \int_{-\infty}^{t_f} dt'\, e^{ip_0t'}J(t',p). \tag{3.6}$$

This is analytic for Im $p_0 < 0$ and so the p_0 integration to be performed by Cauchy's theorem to give

$$t > t_f:\quad \Psi(t,p) = \exp\Big[\big(-ip - \frac{\gamma}{2}\big)t\Big]\widetilde{J}(p - i\frac{\gamma}{2},p). \tag{3.7}$$

The remainder of this section will show that with a current of this type, the simple function given in Eq. (3.2) produces the analogous result but with $\gamma/2$ replaced by $\alpha T[\ln(\omega_p t) + C]$.

B. Saddle point integration

The asymptotic behavior in Eq. (1.1) was obtained in [7, 8, 9, 10] by retaining infrared dominant contributions in the vicinity $p_0 \approx p$. The ansatz in Eq. (3.2) is to be tested in the strip (3.1). To isolate the behavior of the propagator at $p_0 \approx p$, it is necessary to introduce some smearing function $f(p_0)$ that suppresses the effects of large $p_0 - p$. With the smearing, Eq. (3.4) becomes

$$\Psi(t,p) = \int \frac{dp_0}{2\pi} e^{-ip_0 t} s_R(p_0,p) \widetilde{J}(p_0,p) f(p_0).$$

Substituting the ansatz Eq. (3.2) gives

$$\Psi(t,p) = -iN \int \frac{dp_0}{2\pi} e^{\phi(p_0)} \widetilde{J}(p_0,p) f(p_0), \tag{3.8}$$

where

$$\phi(p_0) = -ip_0 t + i\,\frac{p_0 - p}{2\alpha T} + a\exp\Big[i\,\frac{p_0 - p}{\alpha T}\Big]. \tag{3.9}$$

As $t \to \infty$ the integrand oscillates so rapidly that most values of p_0 contribute very little. The dominant contribution comes from the region in which $\phi(p_0)$ is stationary. The first derivative of ϕ is

$$\phi'(p_0) = -it + i\,\frac{1}{2\alpha T} + \frac{ia}{\alpha T}\exp\Big[i\,\frac{p_0 - p}{\alpha T}\Big].$$

The stationary point $\overline{p}_0$ satisfying $\phi'(\overline{p}_0) = 0$ is

$$\overline{p}_0 = p - i\alpha T \ln\left[\frac{\alpha T t - 1/2}{a}\right]. \tag{3.10}$$

An increasing value of t moves the saddle point further down into the lower half of the complex plane. At the saddle point, ϕ itself has the value

$$\phi(\overline{p}_0) = -ipt + (\alpha T t - 1/2)\Big\{ -\ln\left[\frac{\alpha T t - 1/2}{a}\right] + 1\Big\}.$$

The Taylor series expansion of $\phi(p_0)$ about the saddle point is

$$\phi(p_0) = \phi(\overline{p}_0) + \frac{1}{2}\phi''(\overline{p}_0)(p_0 - \overline{p}_0)^2 + \ldots$$

The necessary second derivative is

$$\phi''(\overline{p}_0) = -\left[\frac{\alpha T t - 1/2}{(\alpha T)^2}\right].$$

The fact that the second derivative is real and negative means that the value of $\mathrm{Re}\,[\phi(p_0)]$ is a maximum at the saddle point and that the value decreases if $p_0 - \overline{p}_0$ is positive real or negative real. The dominant contribution to the integral is

$$\begin{aligned}\Psi(t,p) &= -iN\exp[\phi(\overline{p}_0)] \int \frac{dp_0}{2\pi} \widetilde{J}(p_0,p) f(p_0) \\ &\quad \times \exp\Big\{ -\frac{1}{2}(\alpha T t - 1/2)\Big[\frac{p_0 - \overline{p}_0}{\alpha T}\Big]^2\Big\}.\end{aligned}$$

The integration contour can be shifted down below the real p_0 axis. At large time $\alpha T t \gg 1$, if the p_0 contour is a straight line parallel to the real axis keeping the difference $p_0 - \overline{p}_0$ real the integrand falls off very rapidly. The integration yields

$$\Psi(t,p) = -i\frac{N}{\sqrt{2\pi}}\exp[\phi(\overline{p}_0)]\frac{\alpha T}{\sqrt{\alpha T t - 1/2}} \widetilde{J}(\overline{p}_0,p) f(\overline{p}_0).$$

Using the value of $\phi(\overline{p}_0)$ and N gives

$$\begin{aligned}\Psi(t,p) &= -i\exp\Big\{ -ipt - \frac{1}{2} - \alpha T t\Big[\ln\Big[\frac{\alpha T t - 1/2}{a}\Big] - 1\Big]\Big\} \\ &\quad \times \widetilde{J}(\overline{p}_0,p) f(\overline{p}_0).\end{aligned}$$

The relation $\ln(\alpha T/a) = \ln(\omega_p) + C + 1$ is useful in simplifying the result at large time, $\alpha T t \gg 1/2$, and gives the final form

$$\begin{aligned}\Psi(t,p) &= -i\exp\Big\{ -ipt - \alpha T t\Big[\ln[\omega_p t] + C\Big]\Big\} \\ &\quad \times \widetilde{J}(\overline{p}_0,p) f(\overline{p}_0).\end{aligned} \tag{3.11}$$

The time dependence coincides with Eq. (1.1). The stationary point is $\overline{p}_0 = p - i\alpha T[\ln(\omega_p t) + C + 1]$ at large times. This confirms the ansatz.

IV. SOFT AND COLLINEAR PHOTONS

Knowing the electron propagator makes it possible to compute radiatiave processes. For a massive electron in a QED plasma there are various infrared divergences associated with real and virtual photons that eventually cancel. The divergences arise because if an on-shell electron with $P^2 = m^2$ emits an on-shell photon with $K^2 = 0$, the electron subsequently propagates with amplitude

$$\frac{1}{(P-K)^2 - m^2} = \frac{-1}{2|\mathbf{k}|(E - p\cos\theta)}. \tag{4.1}$$

For very soft photons, $|\mathbf{k}| \to 0$, this can produce an infrared divergence. For example, when an electron of momentum P undergoes a hard scattering to momentum P', infrared divergences arise from the real emission and real absorption of soft photons by the incoming electron and by the outgoing electron. Infrared divergences also arise from virtual photons in three ways: from electron self-energy corrections to the ingoing electron and to the outgoing electron; and from virtual photons that link the incoming and outgoing electrons (i.e. corrections to the hard scattering vertex). For massive electrons, the infrared divergences from the real photons exactly cancel those from the virtual photons [13, 14].

If the electron is massless then $E = p$ in Eq. (4.1) and there would also be a collinear singularity arising from $1/(1 - \cos\theta)$. As Blaizot and Iancu showed [8], the thermal propagator for massless electrons does not actually have a pole at $p_0 = p$; it is instead an entire

function of p_0. Since there is no pole, there can be no infrared divergences and no collinear divergences.

However, even though there are no divergences there is still an important quantitative question of how large the radiative corrections are. To compute these corrections, the first step is to compute the vertex that couples a photon to the electron propagator. At this stage it is convenient to return to the strict Bloch-Nordsieck structure of [7, 8, 9] in which the propagating electron has a velocity vector $\hat{v}$. Thus in Eq. (3.2) the combination $p_0 - p$ should be replaced by $p_0 - \hat{v}\cdot\mathbf{p} = P\cdot v$. The Ward-Takahashi identity for a photon of four-momentum K to be radiated by an electron with initial momentum P and final momentum $P-K$ is

$$-K_\mu \Gamma^\mu(P-K,P) = s_R^{-1}(P-K) - s_R^{-1}(P). \quad (4.2)$$

The general solution for the electromagnetic vertex is

$$\Gamma^\mu(P-K,P) = \frac{v^\mu}{K\cdot v}\Big[s_R^{-1}(P) - s_R^{-1}(P-K)\Big]. \quad (4.3)$$

(In the usual case of a free electron $s_R^{-1}(P) = P\cdot v$ so that $\Gamma^\mu(P-K,P) = v^\mu$ as appropriate for the Bloch-Nordsieck approximation.)

Now we employ this in the example of an electron undergoing a hard scattering from P to P'. The initial and final momenta are on shell; $P\cdot v = 0$ and $P'\cdot v' = 0$. During the hard scattering, either the incoming electron or the outgoing electron can emit a real photon with four-momentum K. If $M_{\rm bare}(P',P)$ is the amplitude for the hard scattering without radiation, the amplitude for hard scattering with radiation is approximately

$$M_{\rm bare}(P',P)\epsilon_\mu J^\mu(K).$$

The effective current to which the photon couples is a combination of the electromagnetic vertex above and an off-shell electron propagator:

$$\begin{aligned} J^\mu(K) &= s_R(P-K)\, g\Gamma^\mu(P-K,P) \\ &+ g\Gamma^\mu(P',P'-K)\, s_R(P'-K). \end{aligned}$$

Substituting for the vertex functions gives

$$J^\mu(K) = \frac{gv^\mu}{K\cdot v}\left[\frac{s_R(P-K)}{s_R(P)} - 1\right] + \frac{gv'^\mu}{K\cdot v}\left[1 - \frac{s_R(P'-K)}{s_R(P')}\right]. \quad (4.4)$$

When the momentum transfer $Q^2 = (P-P')^2$ is large, the cross section for hard scattering with the emission of one real photon ($k_0 = k$) is

$$\left[\frac{d\sigma}{dQ^2}\right]_{\rm bare} \int \frac{d^3k}{(2\pi)^3 2k}(1+n)\sum_{\rm pol}|\epsilon_\mu J^\mu(K)|^2, \quad (4.5)$$

where $n = 1/[\exp(\beta k) - 1]$ is the Bose-Einstein function.

For a free electron propagator $s_R^{-1}(P) = 0$ and $s_R^{-1}(P') = 0$ so that the current is

$$J^\mu(K)\Big|_{\rm free} = \frac{-gv^\mu}{k(1-\hat{k}\cdot\hat{v})} + \frac{gv'^\mu}{k(1-\hat{k}\cdot\hat{v}')}.$$

This would produce collinear singularities at $\hat{k}\cdot\hat{v} = 1$ and $\hat{k}\cdot\hat{v}' = 1$ as well as logarithmic and linear infrared divergences from $\int d^3k/k^4$.

However, since the electron propagator is actually an entire function, it has no poles. It is a function of the variable $P\cdot v$. For small values of $K\cdot v$ the current (4.4) becomes

$$\begin{aligned} K\cdot v \ll \alpha T: \quad J^\mu(K)\Big|_{\rm entire} \approx &-g\frac{v^\mu}{s_R}\frac{ds_R}{dp_0}\Big|_{p_0 = \mathbf{p}\cdot\hat{v}} \\ &+g\frac{v'^\mu}{s_R}\frac{ds_R}{dp_0'}\Big|_{p_0' = \mathbf{p}'\cdot\hat{v}'} \end{aligned} \quad (4.6)$$

In the emission cross section (4.5) there is no collinear divergence. The integrand becomes $\int d^3k/k^2$ so there is no infrared divergence. The contribution of soft k to multiplicative factor in Eq. (4.5) is approximately

$$\int^{k_{\rm max}} \frac{d^3k}{(2\pi)^3 2k}\frac{T}{k}\left(\frac{g}{s_R}\frac{ds_R}{dp_0}\right)^2 = \frac{k_{\rm max}}{\alpha T}\frac{(1+2a)^2}{4\pi}.$$

Since $k_{\rm max} \ll \alpha T$ this is a small correction. A similar estimate applies to the effects produced by the absorption of a photon and by the exchange of virtual photons.

V. DISCUSSION

The approximate propagator in Eq. (3.2) has been shown to produce the correct large time behavior in Eq. (1.1). This approximate propagator is the first term of the complete expansion of $s_R(p_0,p)$ that is performed in Appendix B. The approximate propagator Eq. (3.2) has some deficiencies. First, the asymptotic behavior $\ln(\omega_p t)$ is only reached at a very large time t satisfying $\alpha T t \gg 1$, whereas the asymptotic form in Eq. (1.1) sets in at a smaller time $\omega_p t \gg 1$. Second, the corrections to the asymptotic time dependence are of order $1/\alpha T t$ whereas the corrections to Eq. (1.1) are of order $\exp(-\omega_p t)/\omega_p t$.

The absence of any singularity in $s_R(p_0,p)$ eliminates infrared and collinear divergences in radiative processes. However, there are unsettling consequences in higher orders. For example, the contribution to the vacuum polarization tensor, $\Pi^{\mu\nu}(Q)$, of a virtual electron-positron pair would normally have a branch cut that signals the production of a real e^+e^- pair. If $s_R(p_0,p)$ is entire, it appears there can be no e^+e^- production. This situation arises in zero-temperature models of color confinement. There the quark and gluon propagators are thought to have no simple poles and perhaps no singularities at all. A clear example of this occurs in the non-relativistic quark model. There the confining potentials used in the Schroedinger equation produce wavefunctions that fall with distance like $\exp[-f(r)]$, where at large distance $f(r)$ grows more rapidly than r. Form factors and other matrix elements computed with these wavefunctions are entire functions of momentum as a consequence of confinement [15].

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APPENDIX A: COMPLEX TIME AND THE KMS CONDITION

The Kubo, Martin, Schwinger condition [12] requires periodicity in complex time of a certain two point function $\mathcal{S}_>(x)$ defined below. This section shows that the KMS condition is essentially unrelated to the behavior of the retarded propagator in complex time.

1. Definitions

The basic two-point function is

$$\big(\mathcal{S}_>(x)\big)_{\alpha\beta} = -i\mathrm{Tr}\Big[\varrho\,\psi_\alpha(x)\overline{\psi}_\beta(0)\Big],$$

where $\varrho = \exp(-\beta H)/\mathrm{Tr}[\exp(-\beta H)]$ is the density operator. By inserting a complete set of states between the operators, it is easy to show that this is an analytic function of time in the open strip $-\beta < \mathrm{Im}\,(x^0) < 0$. The related function

$$\big(\mathcal{S}_<(x)\big)_{\alpha\beta} = i\mathrm{Tr}\Big[\varrho\,\overline{\psi}_\beta(0)\psi_\alpha(x)\Big]$$

is analytic in the open strip $0 < \mathrm{Im}\,(x^0) < \beta$. The two functions are related by the Kubo, Martin, Schwinger condition [12]

$$\mathrm{Im}\,(x^0) > 0: \quad \mathcal{S}_>(x^0 - i\beta, \mathbf{x}) = -\mathcal{S}_<(x^0, \mathbf{x}). \qquad \text{(A1)}$$

The retarded and advanced propagators are related to the thermal average of the anticommutators:

$$\begin{aligned}\mathcal{S}_R(x) &= \theta(x^0)\Big[\mathcal{S}_>(x) - \mathcal{S}_<(x)\Big]\\ \mathcal{S}_A(x) &= \theta(-x^0)\Big[-\mathcal{S}_>(x) + \mathcal{S}_<(x)\Big].\end{aligned}$$

The only common region of definition for $\mathcal{S}_>(x)$ and $\mathcal{S}_<(x)$ is for x^0 real. The retarded and advanced propagators, with $\theta(\pm x^0)$ omitted, are usually restricted to x^0 real.

2. Momentum space

For massless fermions, the combination of chirality and TCP invariance makes the propagators a linear combination of γ^0 and $\vec{\gamma}\cdot\hat{p}$:

$$\begin{aligned}\mathcal{S}_R(P) &= \frac{1}{2}(\gamma^0 - \gamma\cdot\hat{p})\,s_R(p_0,p)\\ &- \frac{1}{2}(\gamma^0 + \gamma\cdot\hat{p})\,s^*_R(-p^*_0,p)\end{aligned}$$

$$\begin{aligned}\mathcal{S}_A(P) &= \frac{1}{2}(\gamma^0 - \gamma\cdot\hat{p})\,s^*_R(p^*_0,p)\\ &- \frac{1}{2}(\gamma^0 + \gamma\cdot\hat{p})\,s_R(-p_0,p).\end{aligned}$$

The propagators are thus determined by the one function $s_R(p_0,p)$. Knowing this, one can also construct the two-point functions using

$$\mathcal{S}_>(P) = \frac{1}{1+\exp(-\beta p_0)}\Big[\mathcal{S}_R(P) - \mathcal{S}_A(P)\Big] \qquad \text{(A2)}$$

$$\mathcal{S}_<(P) = \frac{-1}{1+\exp(\beta p_0)}\Big[\mathcal{S}_R(P) - \mathcal{S}_A(P)\Big]. \qquad \text{(A3)}$$

Since the retarded propagator is analytic in the upper-half of the complex p_0 plane and the advanced is analytic in the lower-half, generally $\mathcal{S}_>(P)$ and $\mathcal{S}_<(P)$ are not analytic anywhere. Their arguments, p_0, should always be considered as real.

The Fourier transforms of Eq. (A2) and (A3) automatically satisfy the KMS condition (A1). The condition imposes no restriction on the retarded propagator in momentum space, $\mathcal{S}_R(P)$.

APPENDIX B: COMPUTATION OF $\mathbf{s_R(p_0, p)}$

This section computes the Fourier transform of Eq. (1.1) and obtains a complete asymptotic expansion of $s_R(p_0,p)$ valid in the strip

$$-\frac{\pi\alpha T}{2} < \mathrm{Re}(p_0) - p < \frac{\pi\alpha T}{2}. \qquad \text{(B1)}$$

The final result is given in Eqs. (B9) and (B12).

Before beginning this analysis it is useful to examine the full Blaizot and Iancu [7, 8, 9] result for the retarded propagator:

$$s_R(p_0,p) = -i\int_0^\infty dt\; e^{W(p_0,t)} \qquad \text{(B2)}$$

$$W(p_0,t) = i(p_0-p)t - \alpha T t\, f(\omega_p t) \qquad \text{(B3)}$$

$$f(z) = C \;-\gamma + \frac{1-e^{-z}}{z} + \int_0^z ds\,\frac{1-e^{-s}}{s}. \qquad \text{(B4)}$$

The integral that defines $f(z)$ can be computed as a power series:

$$f(z) = C - \gamma + 1 - \sum_{k=1}^{\infty}\frac{(-z)^k}{k\,(k+1)!}.$$

This series converges everywhere in the complex z plane and thus $f(z)$ is an entire function. In particular, $f(z)$ has no branch cuts. However the power series expansion is useless for large z. One can rewrite Eq. (B4) using repeated integration by parts to obtain

$$\begin{aligned}f(z) &= \ln(z) + C + \frac{1}{z} + \frac{e^{-z}}{z}\sum_{k=1}^{n}\frac{(-1)^k k!}{z^k}\\ &+ (-1)^{n+1}(n+1)!\int_z^\infty ds\,\frac{e^{-s}}{s^{n+2}}.\end{aligned}$$

This is also an exact result. Despite the explicit $\ln(z)$, there is no branch point: If z is moved in the complex plane along a counter-clockwise path that encircles the origin, then $\ln(z)$ changes by $2\pi i$ and the integral over s changes by $-2\pi i$. Thus $f(e^{2\pi i}z) = f(z)$: there is no branch cut. For $\mathrm{Re}\, z > 0$ and large, one can use the approximate form

$$f(z) \approx \ln(z) + C.$$

As observed by Blaizot and Iancu [8], the integrand in Eq. (B2) falls so rapidly at large t (large z) that the integral converges any complex value of p_0. Furthermore all the derivatives $\partial^n s_R(p_0,p)/\partial p_0^n$ are finite at any complex value of p_0. Thus $s_R(p_0,p)$ is an entire function of p_0.

1. Exact computation

With the simplified integrand, the Fourier transform to be computed is

$$s_R(p_0,p) = -i\int_0^\infty dt e^{W(p_0,t)}, \tag{B5}$$

where

$$W(p_0,t) = i(p_0-p)t - \alpha T t\big[\ln(\omega_p t) + C\big]. \tag{B6}$$

The computation of the propagator at large t in [7, 8, 9, 10] is most reliable at $p_0 \approx p$. This will be the range in which the Fourier transform is performed. It will be necessary to require that p_0 satisfy Eq. (B1).

The integration contour in Eq. (B5) can be distorted into the complex t plane and the convergence will not be compromised provided that $\mathrm{Re}\, t \to +\infty$ at the terminus of the contour. Define the complex time

$$\overline{t} = \frac{e^{-C-1}}{\omega_p}\exp\Big[i\,\frac{p_0-p}{\alpha T}\Big]. \tag{B7}$$

In the complex t plane, the time derivative of W vanishes at $\overline{t}$. The integration contour in Eq. (B5) can be taken as a straight line from the origin through $\overline{t}$ and since $\mathrm{Re}\,\overline{t} > 0$ the integration converges. The straight line is parametrized by a real variable u such that $t = \overline{t}\,u$. The exponent becomes

$$W(p_0,t) = \overline{W}\big[-u\ln u + u\big],$$

where

$$\overline{W} = a\exp\Big[i\,\frac{p_0-p}{\alpha T}\Big], \tag{B8}$$

and a is given by Eq. (1.3). It will be convenient to express W as

$$W(p_0,t) = \overline{W} - \overline{W}\big[u\ln u - u + 1\big]$$

The quantity in square brackets vanishes at $u=1$ and is otherwise positive throughout the domain $0 \le u \le \infty$. From $u=0$ to $u=1$ it decreases monotonically; from $u=1$ to $u=\infty$ it increases monotonically.

Using $dt = \overline{t}\,du$ and $\overline{t} = \overline{W}/\alpha T$, the retarded propagator can be expressed as

$$s_R(p_0,p) = \frac{-i\overline{W}}{\alpha T}e^{\overline{W}}\Big\{I + J\Big\}, \tag{B9}$$

where the integral has been split into two parts:

$$\begin{aligned} I &= \int_0^1 du\, e^{-\overline{W}[u\ln u - u + 1]} \\ J &= \int_1^\infty du\, e^{-\overline{W}[u\ln u - u + 1]}. \end{aligned}$$

In the first integral change variables from u to x where

$$0 < u < 1: \qquad \frac{1}{2}x^2 = u_<\ln(u_<) - u_< + 1. \tag{B10}$$

As u increases from 0 to 1, x decreases from $\sqrt{2}$ to 0. Expand u in a series

$$u_< = 1 + \sum_{n=1}^\infty (-1)^n c_n x^n.$$

The first few coefficients are $c_1{=}1$, $c_2{=}1/6$, $c_3{=}{-1/72}$, $c_4{=}1/270$. The coefficients satisfy the recursion relation

$$\Big[1 - \frac{n}{2}\Big]c_n = \sum_{k=1}^{n+1} k\, c_k\, c_{n+2-k}.$$

Then I is easily computed

$$\begin{aligned} I &= -\int_0^{\sqrt{2}} dx\,\frac{du_<}{dx}e^{-\overline{W}x^2/2} \\ &= \sum_{n=1}^\infty (-1)^{n+1} n c_n \int_0^{\sqrt{2}} dx\, x^{n-1} e^{-\overline{W}x^2/2}. \end{aligned}$$

In the second integral change variables to y where

$$1 < u < \infty: \qquad \frac{1}{2}y^2 = u_>\ln(u_>) - u_> + 1. \tag{B11}$$

As u increases from 1 to ∞, y increases from 0 to ∞. Expand u in a series

$$u_> = 1 + \sum_{n=1}^\infty c_n y^n,$$

with the same c_n. Then J is

$$\begin{aligned} J &= \int_0^\infty dy\,\frac{du_>}{dy}e^{-\overline{W}y^2/2} \\ &= \sum_{n=1}^\infty n c_n \int_0^\infty dy\, y^{n-1} e^{-\overline{W}y^2/2}. \end{aligned}$$

In the sum $I+J$ there is a partial cancellation. The integrations can be performed exactly with the result

$$I+J= \sum_{k=0}^{\infty}(2k+1)c_{2k+1}\left(-2\frac{d}{d\overline{W}}\right)^k\sqrt{\frac{\pi}{2\overline{W}}}\left[2+\mathrm{erfc}\left(\sqrt{\overline{W}}\right)\right] + \sum_{k=0}^{\infty}(2k+2)c_{2k+2}\left(-2\frac{d}{d\overline{W}}\right)^k\frac{e^{-\overline{W}}}{\overline{W}}. \quad \text{(B12)}$$

The asymptotic behavior of the complementary error function is

$$\mathrm{erfc}\left(\sqrt{\overline{W}}\right)=\frac{e^{-\overline{W}}}{\sqrt{\pi\overline{W}}}\left[1-\frac{1}{2\overline{W}}+\frac{3}{4\overline{W}^2}+\ldots\right]$$

Substitution of Eq. (B12) into Eq. (B9) gives a complete asymptotic expansion for the retarded propagator when p_0 satisfies (B1). Although this expansion is periodic under $p_0 \to p_0+4\pi\alpha T$, such a shift is well outside the domain of validity given in Eq. (B1).

2. Large $\overline{W}$ limit

The asymptotic series (B12) obviously behaves very badly at small values of $\overline{W}$. The most useful application of the results is clearly when $\mathrm{Re}\,\overline{W}^{1/2}$ is large and positive. In this region the exponentials are negligible

$$\begin{aligned} s_R(p_0,p) &= -i\frac{\sqrt{2\pi\overline{W}}}{\alpha T}e^{\overline{W}}\sum_{k=0}^{\infty}c_{2k+1}\frac{(2k+1)!!}{\overline{W}^k} \quad \text{(B13)}\\ &= -i\frac{\sqrt{2\pi\overline{W}}}{\alpha T}e^{\overline{W}}\left[1-\frac{24}{\overline{W}}-\frac{23}{1152\overline{W}^2}+\ldots\right]. \end{aligned}$$

The simplest approximation is to keep only the first term in Eq. (B13). In terms of p_0 and p the propagator is

$$s_R(p_0,p)=-i\frac{\sqrt{2\pi a}}{\alpha T}\exp\left[i\,\frac{p_0-p}{2\alpha T}+a\exp\left[i\,\frac{p_0-p}{\alpha T}\right]\right]. \quad \text{(B14)}$$

This is analytic for any finite p_0; it vanishes in the limit Im $p_0 \to +\infty$; and it diverges when $\mathrm{Re}\,p_0=p$ and Im $p_0 \to -\infty$ in agreement with Eq. (1.2). This is the form used throughout the paper.

Before concluding, it is necessary to discuss how $\overline{W}$ can be large. From Eq. (B8) the absolute value is

$$\left|\overline{W}\right| = ae^{-\mathrm{Im}\,p_0/\alpha T}=\frac{3g}{4\pi}\exp\left[-C-1-\frac{\mathrm{Im}\,p_0}{\alpha T}\right] \quad \text{(B15)}$$

For small g there are two ways in which the magnitude $|\overline{W}|$ can be large. One possibility is to restrict consideration to the region in which $\mathrm{Im}\,p_0/\alpha T$ is large and negative. This is perfectly valid, but it excludes the physically important region of real p_0. The other possibility is for the constant C to be large and negative.

Both groups [9, 10] showed that value of C in the Coulomb gauge is $C=\ln\sqrt{3}+\gamma_E-1=0.12652\ldots$ In covariant gauges the value of C can be different. When the zero temperature, Feynman propagator for the photon is

$$D_F^{\mu\nu}(q)=\frac{1}{q^2+i\epsilon}\left[-g^{\mu\nu}+(1-\lambda)\frac{q^\mu q^\nu}{q^2}\right],$$

it is necessary to impose an infrared cutoff μ. Blaizot and Iancu [9] discuss two limits: $\mu t \ll 1$ and $\mu t \gg 1$. If $\mu t \gg 1$ then C is independent of the gauge parameter λ. However in the calculation of Sec. B1, the dominant contribution comes from finite values of t. (The integral I sums t in the range $0 \le t \le \bar{t}$. If the integral limits on J are changed to $0 \le y \le \sqrt{2}$, corresponding to $\bar{t} \le t \le e\bar{t}$, the error made is $2.35 \times e^{-\overline{W}}/\overline{W}$, which is negligible.) Since the important values of t are finite in the previous integration, then as the infrared cutoff μ is reduced the appropriate limit is $\mu t \ll 1$. In this limit Blaizot and Iancu [9] showed that C does depend on the gauge parameter λ:

$$C=\frac{\lambda}{2}+\ln\sqrt{3}+\gamma_E-1=\frac{\lambda}{2}+0.12652\ldots \quad \text{(B16)}$$

Consequently a large, negative λ will make C large and negative, which will make $|\overline{W}|$ large thus justifying the approximation. In [16] the range $-857 < \lambda < 433$ was explored.

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